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REACTION OF A PIEZOCERAMIC SHELL TO CONCENTRATED EFFECTS*

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Equations are derived for the theory of shallow piezoceramic shells polarized along the generators. These equations are used to construct the Green's matrix and to investigate the reaction of an infinite and a finite shell to concentrated effects. The asymptotic is written down for the mechanical forces and moments as well as for the electrical field potential in the neighborhood of the point of application of the concentrated functional.

Equations for piezoceramic shells of revolution have been obtained in /1/. Application of the asymptotic method /2/ to the derivation of the equations for the theory of piezoceramic shells is examined in /3/.

1. Equations of a piezoceramic shell in displacements. Let a closed shallow piezoceramic shell be referred to an orthogonal coordinate system α, β, γ (α, β coincide with the lines of principal curvature). The shell is polarized along the axis α. In this case the equations of state have the form /4/

$$\sigma_\alpha = c_{33}\varepsilon_\alpha + c_{13}\varepsilon_\beta - e_{33}E_\alpha,\ \tau_{\alpha\beta} = c_{44}\varepsilon_{\alpha\beta} - e_{15}E_\beta \tag{1.1}$$
$$\sigma_\beta = c_{13}\varepsilon_\alpha + c_{11}\varepsilon_\beta - e_{13}E_\alpha,\ D_\beta = e_{15}\varepsilon_{\alpha\beta} + \varepsilon_{11}E_\beta$$
$$D_\alpha = e_{33}\varepsilon_\alpha + e_{13}\varepsilon_\beta + \varepsilon_{33}E_\alpha,\ D_\gamma = \varepsilon_{11}E_\gamma$$

Here $\sigma_\alpha, \sigma_\beta, \tau_{\alpha\beta}, \varepsilon_\alpha, \varepsilon_\beta, \varepsilon_{\alpha\beta}$ are components of the stress and strain tensors, $E_\alpha, E_\beta, E_\gamma, D_\alpha, D_\beta, D_\gamma$ are the coresponding vector components of the electrical field intensity and the induction, c_{ik} are elastic, e_{ik} are piezoelectric constants, and $\varepsilon_{11}, \varepsilon_{33}$ are the dielectric permittivities of the medium.

Making use of the usual relationships of the theory of very shallow shells /2,5/ as well as of the Maxwell equations for dielectrics $\operatorname{div} \bar D = 0$, $\bar E = -\operatorname{grad} \varphi$ (φ is the field potential), we arrive at the system of equations in displacements

$$L_{ij}u_j = P_i,\ L_{ij} = L_{ji}\ (i, j = 1, 2, 3, 4) \tag{1.2}$$
$$L_{11} = h\left(\frac{c_{33}}{A^2}\partial_1^2 + \frac{c_{44}}{B^2}\partial_2^2\right),\quad L_{12} = \frac{h}{AB}(c_{13} + c_{44})\partial_1\partial_2$$
$$L_{13} = \frac{h}{A}(k_1c_{33} + k_2c_{13})\partial_1,\quad L_{23} = \frac{h}{B}(k_1c_{13} + k_2C_{11})\partial_2$$
$$L_{14} = h\left(\frac{e_{33}}{A^2}\partial_1^2 + \frac{e_{15}}{B^2}\partial_2^2\right),\quad L_{24} = \frac{h}{AB}(e_{13} + e_{15})\partial_1\partial_2$$
$$L_{34} = \frac{h}{A}(k_1e_{33} + k_2e_{13})\partial_1,\quad L_{22} = h\left(\frac{c_{44}}{A^2}\partial_1^2 + \frac{c_{11}}{B^2}\partial_2^2\right)$$
$$L_{33} = h(k_1^2c_{33} + 2k_1k_2c_{13} + k_2^2c_{11}) + \frac{h^3c_{33}}{12A^4}\partial_1^4 + \frac{h^3c_{11}}{12B^4}\partial_2^4 +$$
$$\frac{h^3(2c_{13} + 4c_{44})}{12A^2B^2}\partial_1^2\partial_2^2,\quad L_{44} = -h\left(\frac{\varepsilon_{33}}{A^2}\partial_1^2 + \frac{\varepsilon_{11}}{B^2}\partial_2^2\right)$$
$$\partial_1 = \partial/\partial\alpha,\quad \partial_2 = \partial/\partial\beta,\quad \partial_1^i\partial_2^j = \frac{\partial^{i+j}}{\partial\alpha^i\,\partial\beta^j}$$

Here $u_j\ (j = 1, 2, 3)$ are components of the displacement vector, $u_4 = \varphi$, $P_i\ (i = 1, 2, 3)$ are components of the surface load vector, $P_4 = 0$, h, $k_1 = R_1^{-1}$, $k_2 = R_2^{-1}$ are the shell thickness and principal shell curvatures, and A and B are the coefficients of the first quadratic form (for simplicity we later consider a cylindrical shell $R_1 = \infty$, and in addition we set $A = B = R_2$).

It was assumed in the derivation of (1.2) that the shell is in a vacuum and its surfaces are not electrified. In this case it is possible to set $D_\gamma = E_\gamma = 0$, $\varphi = \varphi(\alpha, \beta)$.

2. Fundamental solution of the system (1.2). At the points α_0, $\beta_0 + mT$ ($m = 0.1, \ldots, \omega - 1$), let a T-periodic system of concentrated forces with components $P_\alpha = P_1$, $P_\beta = P_2$, $P_\gamma = P_3$ be applied to the shell (Fig.1).

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In this case we represent the solution of the system in the form

$$u_i = A_{ij}\Psi_j \ (i, j = 1, 2, 3, 4) \tag{2.1}$$

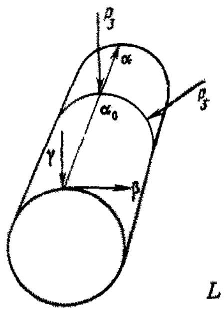

Fig.1

where A_{ij} are the corresponding cofactors of the matrix elements of the differential operators $\|L_{ij}\|$, while the functions Ψ_j are related to the fundamental solution $E(\alpha, \gamma)$ by

$$\Psi_j = -\frac{P_j E(\alpha, \beta)}{F_1 R_2^2 a_0}, \quad F_1 = h^6/R_2^{10} \tag{2.2}$$
$$a_0 = -c_{11}^2 g/12, \ g = c_{44}\varepsilon_{11} + e_{15}^2$$

The T-periodic fundamental solution $E(\alpha, \beta)$ is determined from the equation

$$L(\partial_1, \partial_2) E(\alpha, \beta) = \delta(\alpha) \sum_{k=-\infty}^{\infty} \delta(\beta - kT) \tag{2.3}$$

$$L(\partial_1, \partial_2) = L_0(\partial_1, \partial_2) + \sum_{j=0}^{3} \left(\frac{R_2}{h}\right)^2 \frac{a_j'}{a_0} \partial_1^{2j}\partial_2^{8-2j}$$

$$L_0(\partial_1, \partial_2) = \sum_{i=0}^{5} \frac{a_i}{a_0} \partial_1^{2i}\partial_2^{10-2i}$$

$$a_1 = \frac{c_{11}}{12}[-c_{11}l + \varepsilon_{11}(c_{13}^2 - 3c_{44}^2) + 2e_{15}\eta - c_{44}(e_{13}^2 + 4e_{15}^2)]$$

$$a_2 = \frac{1}{12}[\theta q + 2rs(c_{11}e_{33} + 2e_{15}f) - t(r^2 + e_{15}^2) - c_{11}(2c_{44}\xi -$$
$$2c_{44}f\varepsilon_{33} - 2e_{33}e_{15}\lambda - e_{33}^2 c_{11})], \quad a_3 = \frac{1}{12}[-c_{33}c_{44}(e_{15}^2 + c_{11}\varepsilon_{33} +$$
$$\theta) + \xi q + 2rs(2e_{33}f + c_{33}e_{15}) - \lambda(c_{33}r^2 + c_{11}e_{33}^2) - 2e_{33}e_{15}t],$$

$$a_4 = \frac{c_{33}}{12}[\varepsilon_{33}q - c_{44}\xi + 2e_{33}(e_{13}s + e_{15}c_{13}) - c_{33}r^2 - c_{11}e_{33}^2] -$$
$$\frac{1}{6}c_{44}e_{33}^2 f, \quad a_5 = -\frac{1}{12}c_{33}c_{44}\mu$$

$$a_0' = a_1' = 0, \ a_2' = (c_{13}^2 - c_{11}c_{33})g, \ a_3' = c_{44}[\varepsilon_{33}(c_{13}^2 -$$
$$c_{11}c_{33}) + e_{13}(e_{33}c_{13} - e_{13}c_{33}) + e_{33}\eta], \ f = c_{13} + 2c_{44}, \ q =$$
$$c_{13}f - c_{11}c_{33}$$

$$r = e_{13} + e_{15}, \ s = c_{13} + c_{44}, \ t = c_{11}c_{33} + 2c_{44}f,$$
$$\lambda = c_{44} + 2f$$
$$\eta = c_{13}e_{13} - c_{11}e_{33}, \ \theta = c_{11}\varepsilon_{33} + 2\varepsilon_{11}f, \ \xi = \varepsilon_{11}c_{33} + 2\varepsilon_{33}f$$
$$e = c_{44}\varepsilon_{33} + c_{33}\varepsilon_{11}, \ \mu = \varepsilon_{33}c_{33} + e_{33}^2$$

Using the procedure of /6–9/, we find the fundamental solution in the form of a superposition of exponential functions of their complex variables

$$E(\alpha - \alpha_0, \beta - \beta_0) = \frac{1}{T}C_0(\alpha - \alpha_0) + \tag{2.4}$$
$$\frac{2}{T}\operatorname{Re}\sum_{\nu=1}^{5}\sum_{k=1}^{\infty} \frac{\exp[ik\omega(\zeta_\nu^{(k)} - \zeta_{\nu 0}^{(k)})\operatorname{sgn}(\alpha - \alpha_0)]}{(ik\omega)^8 \Delta_k'(z_\nu^{(k)})}$$

$$2C_0(\alpha) = \left(\frac{h}{R_2}\right)^3 \left\{\frac{a_0|\alpha|^5}{120a_3'} - \left(\frac{h}{R_2}\right)^2 \frac{a_0 a_5}{(a_3')^2}\left[|\alpha| - \operatorname{Im}\frac{\exp(i|\alpha|z_0)}{z_0}\right]\right\}$$

$$\zeta_\nu^{(k)} = \beta + z_\nu^{(k)}\alpha, \ \zeta_{\nu 0}^{(k)} = \beta_0 + z_\nu^{(k)}\alpha_0, \ \operatorname{Im} z_\nu^{(k)} > 0$$

$$\Delta_k'(z) = \frac{d}{dz}[\Delta_k(z)], \quad \Delta_k(z) = -\frac{1}{(k\omega)^{10}}L(z, 1) \quad (k = 1, 2, \ldots)$$

$$z_0 = [(R_2/h)^2(a_3'/a_5)]^{1/4}(1 + i)$$

The expression $L(x, y)$ is defined in (2.3). For shells from the piezoceramics *PZT*-4, *PZT*-5 and certain others, the quantities z_ν are simple roots of the characteristic polynomial $\Delta_k(z)$ for each fixed k.

3. The principal part of the fundamental solution. It agrees with the fundamental solution of the operator $L_0(\partial_1, \partial_2)$ and has the form

$$E_0(\alpha - \alpha_0, \beta - \beta_0) = \frac{1}{T}g_0(\alpha - \alpha_0) + \tag{3.1}$$

$$\frac{2}{T}\operatorname{Re}\sum_{\nu=1}^{5}\sum_{k=1}^{\infty}\frac{\exp\left[ik\omega\left(\zeta_\nu-\zeta_{\nu0}\right)\operatorname{sgn}\left(\alpha-\alpha_0\right)\right]}{(ik\omega)^9\,\Delta'\left(z_\nu\right)}$$

$$g_0(\alpha)=\frac{a_0\,|\alpha|^9}{2\cdot 9!\,a_5},\quad \zeta_\nu=\beta+z_\nu\alpha,\quad \zeta_{\nu0}=\beta_0+z_\nu\alpha_0,\quad \operatorname{Im} z_\nu>0$$

Here z_ν are roots of the characteristic polynomial

$$\Delta(z)=L_0(z,1)=\sum_{j=0}^{5}\frac{a_j}{a_0}z^{2j}$$

It can be shown that the following relations hold that are valid for any natural k:

$$2\operatorname{Re}\sum_{\nu=}^{5}\frac{\left(z_\nu^{(k)}\right)^q}{\Delta_k'\left(z_\nu^{(k)}\right)}=\begin{cases}0, & q=0,1,2,\ldots,8\\ \frac{a_0}{a_5}, & q=9\end{cases} \tag{3.2}$$

By using (3.2) closed expressions can be obtained for the higher derivatives of $E_0(\alpha,\beta)$. For instance

$$\partial_1^{9-j}\partial_2^{j}E_0(\alpha,\beta)=\frac{1}{T}\left[-\operatorname{Im}\sum_{\nu=1}^{5}\frac{z_\nu^{9-j}}{\Delta'(z_\nu)}\operatorname{ctg}\frac{\omega\zeta_\nu}{2}\right] \tag{3.3}$$

4. Green's matrix for a finite piezoceramic shell.

We represent the components of the Green's matrix in the form (no summation over repeated subscripts)

$$u_{ij}(\alpha,\beta,\alpha_0,\beta_0)=u_{ij}^0(\alpha,\alpha_0)+A_{ij}\Psi_j \tag{4.1}$$

$$\Psi_j=-\frac{P_j}{F_1a_0R_2^2}G(\alpha,\beta,\alpha_0,\beta_0)$$

$$G(\alpha,\beta,\alpha_0,\beta_0)=\frac{2}{T}\operatorname{Re}\sum_{k=1}^{\infty}\sum_{\nu=1}^{10}B_\nu^{(k)}\exp\left[ik\omega\zeta_\nu^{(k)}\right]+E(\alpha-\alpha_0,\beta-\beta_0)$$

Here $u_{ij}^0(\alpha,\alpha_0)$ is the general solution of the system (1.2) for $P_j=0$ $(j=1,2,3,4)$, $B_\nu^{(k)}$ are constants determined from the boundary conditions on the shell endfaces, $E(\alpha,\beta)$ is the fundamental solution of (2.4), $u_{ij}(\alpha,\beta,\alpha_0,\beta_0)$ is the displacement $(i=1,2,3)$ and potential of the electrical field $(i=4)$ at the point (α,β) due to the action of the concentrated force $P_j(j=1,2,3)$ or the charge P_4 at the point (α_0,β_0). The second component in the expression for the function G is the regular solution of the homogeneous equation (2.3).

The forces and moments in the shell as well as the intensity vector and the electrical induction vector are expressed in terms of the Green's matrix components (4.1) by using (1.1), geometric relationships, and Maxwell's equations.

The representations (4.2) afford the possibility of satisfying four mechanical and one electrical boundary condition on each of the endfaces because of the slection of the constants $B_\nu^{(k)}$ and the functions $u_{ij}^0(\alpha,\alpha_0)$.

5. Action of radial concentrated forces on a cylindrical piezoceramic shell.

Let us consider a piezoceramic shell finite in α, and closed in β and loaded at the points $\alpha=\alpha_0$, $\beta=\beta_0+mT$ $(m=0,1,\ldots,\omega-1)$ by a T-periodic system of radial concentrated forces. We assume that the shell is under moving-hinge clamping conditions, while the endfaces are covered by grouned electrodes. Then the mechanical and electrical boundary conditions on the endfaces $\alpha=0$ and $\alpha=l_0$ take the form

$$T_1=M_1=v=w=\varphi=0 \tag{5.1}$$

According to (1.2), (2.1) and (4.1) we have

$$u_{33}=w(\alpha,\beta)=A_{33}\Psi_3+u_{33}^{\circ} \tag{5.2}$$

$$u_{43}=\varphi(\alpha,\beta)=A_{43}\Psi_3+u_{43}^{\circ}$$

$$A_{33}=\frac{h^3}{R_2^6}\sum_{j=0}^{3}b_j^{33}\partial_1^{6-2j}\partial_2^{2j},\quad A_{43}=\frac{h^3}{R_2^6}\sum_{j=0}^{2}b_j^{43}\partial_1^{5-2j}\partial_2^{2j}$$

$$b_0^{33}=-c_{44}\mu,\quad b_1^{33}=[-c_{33}m-c_{44}^2\varepsilon_{33}+2e_{33}rs-e_{33}^2c_{11}-2e_{15}e_{33}c_{44}+\varepsilon_{33}s^2-c_{33}r^2],\quad b_2^{33}=[-c_{11}l-\varepsilon_{11}c_{44}^2+2e_{15}rs-e_{15}^2c_{44}-2e_{33}e_{15}c_{11}+\varepsilon_{11}s^2-c_{44}r^2],\quad b_3^{33}=-c_{11}g$$

$$b_0^{43} = c_{44}\rho, \quad b_1^{43} = e_{15}(c_{11}c_{33} - c_{13}^2) + c_{44}\eta, \quad b_2^{43} = 0$$
$$m = \varepsilon_{33}c_{11} + \varepsilon_{11}c_{44}, \quad \rho = e_{33}c_{13} - e_{13}c_{33}$$
$$u_{33}^\circ = \sum_{\nu=1}^{4} B_\nu^\circ \exp(z_\nu^\circ \alpha) - \frac{12a_5\rho}{c_{33}a_3'\mu} B_5^\circ$$
$$u_{43}^\circ = \sum_{\nu=1}^{4} B_\nu^\circ \left(-\frac{\rho}{\mu}\right) \frac{\exp(z_\nu^\circ\alpha)}{z_\nu^\circ} + \left(\frac{12a_5\rho^2}{c_{33}a_3'\mu^2} - \frac{c_{33}}{\mu}\right)\alpha B_5^\circ + B_6^\circ$$
$$z_1^\circ = z_0, \quad z_2^\circ = \bar{z}_0, \quad z_3^\circ = iz_0, \quad z_4^\circ = -z_0$$

where B_ν° $(\nu = 1, 2, \ldots, 6)$ are arbitrary constants.

Results of computing the quantities

$$\langle w\rangle = \frac{w(\alpha, \beta)}{\Omega}, \quad \langle\varphi\rangle = \frac{1}{V_0}\int_0^T \varphi(\alpha, \beta)\, d\beta \qquad (5.3)$$
$$\Omega = -\frac{P_3 b_0^{33}}{Tha_3'}, \quad V_0 = -\frac{P_3 b_0^{43}}{ha_3'}$$

along the coordinate α are represented in Figs. 2 and 3 for a cylindrical shell from the piezoceramic *PZT*-5 /4/ loaded by radial concentrated forces P_3 for $R_2/h = 50, \omega = 6$ and relative length of the shell $l_0 = L/R_2 = 1$. The curves *1* and *2* are constructed for $\beta_0 = 0$, $\alpha_0 = 0{,}5l_0$ and $0.25l_0$, respectively. The dashed curve corresponds to an infinite shell for the same values of the parameters.

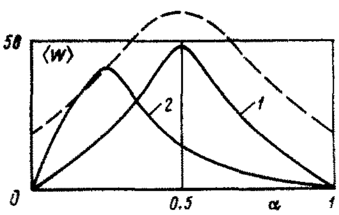

Fig.2

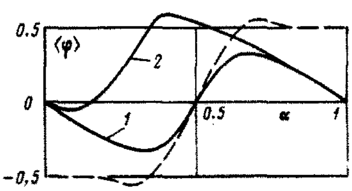

Fig.3

For $l_0 \geqslant 2$ the deflections in a finite shell subjected to radial forces applied at the points $\alpha_0 = 0.5$, $\beta_0 = mT$ $(m = 0, 1, \ldots, \omega - 1)$ are practically in agreement with the corresponding deflections in an infinite shell. The average potential $\langle\varphi\rangle$ equals zero at the endfaces for a finite shell, while it asymptotically approaches the lines $\langle\varphi\rangle = \pm 0.5$ according to the law

$$\langle\varphi\rangle = 0.5 \,\mathrm{sgn}\,\alpha\, \{1 - \mathrm{Im}\,[i \exp(iz_0\alpha)]\}$$

for an infinite shell.

The asymptotic values of the bending moments and transverse forces in the neighborhood of the point of application of the concentrated functional have the form

$$M_k = \varkappa\, \mathrm{Im} \sum_{\nu=1}^{5} d_\nu m_\nu^{(k)} \Phi_1(\zeta_\nu) \quad (k = 1, 2, 3), \quad M_3 = H \qquad (5.4)$$
$$N_k = \varkappa\, \mathrm{Im} \sum_{\nu=1}^{5} d_\nu n_\nu^{(k)} \Phi_2(\zeta_\nu) \quad (k = 1, 2)$$
$$\varkappa = -P_3/6Ta_0, \quad d_\nu = \Big(\sum_{j=0}^{3} b_j^{33} z_\nu^{6-2j}\Big)/\Delta'(z_\nu)$$
$$m_\nu^{(1)} = (c_{33}z_\nu^2 + c_{13})/\omega, \quad m_\nu^{(2)} = (c_{13}z_\nu^2 + c_{11})/\omega$$
$$m_\nu^{(3)} = 2c_{44}z_\nu/\omega, \quad n_\nu^{(1)} = z_\nu(2c_{44} + c_{13} + c_{33}z_\nu^2)/2R_2$$
$$n_\nu^{(2)} = (c_{13}z_\nu^2 + c_{44}z_\nu + c_{11})/2R_2$$
$$\Phi_1(\zeta_\nu) = \ln\left(2\sin\frac{\omega\zeta_\nu\,\mathrm{sgn}\,\alpha}{2}\right)$$
$$\Phi_2(\zeta_\nu) = \mathrm{ctg}\,\frac{\omega\zeta_\nu}{2}$$

The remaining mechanical, as well as electrical quantities, are bounded.

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